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Editor:

R. VAIDYANATHASWAMY, M.A., D.Sc.

Joint Editor:

A. NARASINGA RAO, M.A., L.T.

Collaborators:

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S. C. DHAR, D.Sc.; S. R. U. SAVOOR, M.A., D.Sc.;
V. RAMASWAMI AIYAR, M.A.; T. SURYANARAYANA, M.A.

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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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Results (20, 21) of § 8, can now be written in the form

$$a_{n+k} + (-1)^k a_{n-k} = a_n M_k \qquad \ldots \quad (24)$$

$$a_{n+k} - (-1)^k a_{n-k} = A_n N_k. \qquad \ldots \quad (25)$$

∴ $$a^2_{n+k} - a^2_{n-k} = a_n A_n N_k M_k \qquad \ldots \quad (26)$$

and $$2a_{n+k} = a_n M_k + A_n N_k. \qquad \ldots \quad (27)$$

∴ $a_n M_k$, $a_{n+k}$, $A_n N_k$ are in A.P.

From (24, 25), we further get

$$N_{n+k} + (-1)^k N_{n-k} = N_n M_k. \qquad \ldots \quad (28)$$

∴ $$N_{2k} = N_k M_k \qquad \ldots \quad (29)$$

and $$M_{n+k} + (-1)^k M_{n-k} = M_n M_k. \qquad \ldots \quad (30)$$

Again $$N_{n+k} - (-1)^k N_{n-k} = M_n N_k \qquad \ldots \quad (31)$$

and $$M_{n+k} - (-1)^k M_{n-k} = [M_{n+1} + M_{n-1}] N_k$$

$$= 5 N_n N_k. \qquad \ldots \quad (32)$$

**10.** To prove

$$\begin{vmatrix} A_p & A_q \\ B_p & B_q \end{vmatrix} = -5 \begin{vmatrix} a_p & a_q \\ b_p & b_q \end{vmatrix}$$

We have $[A_p ; B_q] = [a_{p+1} + a_{p-1} ; b_{q+1} + b_{q-1}]$

$$= [a_{p+1} ; b_{q+1}] + [a_{p+1} ; b_{q-1}] + [a_{p-1} ; b_{q+1}] + [a_{p-1} ; b_{q-1}]$$

$$= \{ (-1)^q N_{p-q} + (-1)^{q-2} N_{p-q+2} + (-1)^q N_{p-q-2} + (-1)^{q-2} N_{p-q} \} [a_2 ; b_1]$$

$$= (-1)^q [M_2 + 2] N_{p-q} . [a_2 ; b_1] = (-1)^q 5 N_{p-q} . [a_2 ; b_1].$$



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But $[a_p; b_q] = (-1)^{q-1} . N_{p-q} . [a_2 ; b_1]$.

$$\therefore \quad [A_p ; B_q] = -5 . [a_p ; b_q]. \quad \ldots \quad (33)$$

In particular $\begin{vmatrix} A_2 & A_1 \\ A_3 & A_2 \end{vmatrix} = -5 \begin{vmatrix} a_2 & a_1 \\ a_3 & a_2 \end{vmatrix}$

Therefore, the medial difference of the auxiliary series is $-5$ times the medial difference of the original series.

**11.** We have $\quad a_{n+k} + (-1)^k a_{n-k} = a_n M_k$.

$$\therefore \quad a_{rk} + (-1)^k a_{(r-2)k} = a_{(r-1)k} M_k. \quad \ldots \quad (34)$$

Let $S_n = a_k + a_{2k} + a_{3k} + a_{4k} + \ldots\ldots + a_{nk}$.

Hence we obtain,

$$S_n \{ 1 - M_k + (-1)^k \} = a_k + (-1)^{k-1} a_0 - a_{(n+1)k} - (-1)^{k+1} a_{nk}.$$

Therefore

$$S_n = \frac{\{ a_k - a_{(n+1)k} \} + (-1)^{k+1} \{ a_0 - a_{nk} \}}{1 + (-1)^k - M_k} = \Sigma a_{nk}. \quad (35)$$

In particular,

$$\Sigma a_{2n} = - \{ a_2 - a_{2n+2} - a_0 + a_{2n} \}$$

$$= a_{2n+1} - a_1.$$

**12.** From [4], we have

$$M_p = N_{p-q+1} M_q + N_{p-q} M_{q-1},$$

$$M_q = N_{p-q+1} M_{2q-p} + N_{p-q} M_{2q-p-1}.$$

$$\therefore \quad \{M_p, M_q\} = \{M_q, N_{p-q}\}$$

$$= \{M_q, M_{2q-p} = \{M_q, M_{p-2q}\}$$

$$= \{M_q, M_{p-4q}\} = \ldots\ldots$$

$$= \{M_q, M_{p-2rq}\}. \qquad \ldots \quad (36)$$

**13.** To find the conditions under which $M_p$ shall be prime to every preceding term of the series.

Let $M_q$ be the least term not prime to $M_p$.

Let $\qquad p = 2rq \pm s, \quad s \leqslant q.$

Then, $\qquad \{M_p, M_q\} = \{M_q, M_s\}.$

$\therefore \qquad s \nless q.$

$\therefore \qquad s = q.$

$\therefore$ $p$ is an odd multiple of $q$.

$\therefore$ $M_p$ will be prime to every preceding term of the series if $p$ has no odd factor other than unity or itself.

$\therefore$ $p$ is either a prime or of the form $2^n$.

**14.** In result [28] of § 9, put $n = (r-1)k$;

then $$N_{rk} = N_{(r-1)k} \cdot M_k - (-1)^k N_{(r-2)k}. \qquad \ldots \quad (37)$$

Putting $r = 2, 3, 4, 5, \ldots$ in succession, we obtain

$N_{2k} = N_k M_k. \qquad$ (*cf.* 54).

$N_{3k} = N_{2k} M_k - (-1)^k N_k = N_k \{M^2_k - (-1)^k\}.$

$N_{4k} = N_{3k} M_k - (-1)^k N_{2k} = N_k \{M^3_k - 2(-1)^k M_k\}.$

$N_{5k} = N_{4k} M_k - (-1)^k N_{3k} = N_k \{M^4_k - 3(-1)^k M^2_k + 1\}.$

From this, we can easily prove by Mathematical Induction that

$$N_{(r+1)k} = N_k \{ {}^{r}C_0 M^r_k - (-1)^k \, {}^{r-1}C_1 . M_k^{r-2} + (-1)^{2k} \, {}^{r-2}C_2 . M_k^{r-4} - \ldots\ldots \}$$

$$= N_k . \sum_{s=0}^{r/2} {}^{r-s}C_s . (-1)^{(k+1)s} . M_k^{r-2s}. \qquad \ldots (38)$$

$$\therefore \; N_{r+1} = \sum_{s=0}^{r/2} {}^{r-s}C_s.$$

In result [30] of § 9, put $n = (r-1)k$,

then $$M_{rk} = M_{(r-1)k} M_k + (-1)^{k+1} M_{(r-2)k}. \qquad \ldots (39)$$

From this we obtain,

$$M_{rk} = M_k^r + \sum_{s=1}^{r/2} (-1)^{s(k+1)} \{ {}^{r-s}C_s + {}^{r-s-1}C_{s-1} \} M_k^{r-2s} \qquad \ldots (40)$$

$$= \sum_{s=0}^{r/2} (-1)^{s(k+1)} . \frac{r \, | \, r-s-1}{s \, | \, r-2s} . M_k^{r-2s}. \qquad \ldots (41)$$

In result (31) of § 9, put $n = rk$,

then $$N_{(r+1)k} = N_k M_{rk} + (-1)^k N_{(r-1)k}. \qquad \ldots (42)$$

From this, we get

$$N_{rk} = N_k \{ M_{(r-1)k} + (-1)^k M_{(r-3)k} + (-1)^{2k} M_{(r-5)k} + \ldots \}. \qquad (43)$$

The last term in the bracket is $(-1)^{(\frac{r}{2}-1)k} M_k$ or $(-1)^{\frac{r-1}{2}k}$ according as $r$ is even or odd.

Further $N_{r+1}$ is the co-efficient of $x^r$ in the expansion of

$$\frac{1}{1-x-x^2} = 1 + x(1+x) + x^2(1+x)^2 + x^3(1+x)^3 + \ldots + x^n(1+x)^n + \ldots$$

Now the co-efficient of $x^r$ in the expansion of $x^n(1+x)^n$, is ${}^nC_{r-n}$.

$$\therefore \qquad N_{r+1} = \Sigma\, {}^nC_{r-n}, \qquad \tfrac{r}{2} \leqslant n \leqslant r.$$

## Summary of Results.

§ 1. Definition: Every term of the series is the sum of the preceding two.

§ 2. General term: $a_{r+1} = N_r\, a_2 + N_{r-1}\, a_1$,

where $N_r$ is the $r$th term of the series 1, 1, 2, 3, 5, 8, 13,... ...

Summation formula: $\sum\limits_{n=1}^{m} a_n = a_{m+2} - a_2$.

§ 3. $$N^2_r + N^2_{r-1} = N_{2r-1}.$$

Proof of the theorem: If $n \equiv 0$ [mod. $m$], then $N_n \equiv 0$ [mod. $N_m$].

§ 4. If the H. C. F. of $p$, $q$, denoted by the symbol $\{p, q\}$, be $h$

then $$\{N_p, N_q\} = N_h,$$

so that $\{N_p, N_q\} = 1$, if $\{p, q\} = 1$ or 2;

and $\{N_p, N_q\} = 1$, $q = 1, 2, 3, \ldots, p-1$,

if $p$ is a prime.

§ 5. $$\underset{n \to \infty}{\mathrm{Lt}} \frac{a_{n-1}}{a_n} = \frac{\sqrt{5}-1}{2}.$$

The reason why it is called a 'Medial' Progression.

§ 6. 'Negative' terms of a M. P.

§ 7. Every three terms $a_p$, $a_q$, $a_r$ satisfy the relation

$$\Sigma\,(-1)^q\, a_r\, N_{p-q} = 0.$$

Hence given any two, any other can be found.

§ 8. $$\begin{vmatrix} a_{p+m} & a_{q+m} \\ b_p & b_q \end{vmatrix} = (-1)^{q-1}\, N_{p-q} \begin{vmatrix} a_{m+2} & a_{m+1} \\ b_2 & b_1 \end{vmatrix}$$

where $\Sigma\, a_n$ and $\Sigma\, b_n$ are two series in M. P.

§ 9. Auxiliary Series of $\Sigma\, a_n$, defined by the relation

$$A_n = a_{n+1} + a_{n-1};$$

$$M_n = N_{n+1} + N_{n-1}.$$

Some formulae concerning them.

§ 10.
$$\begin{vmatrix} A_p & A_q \\ B_p & B_q \end{vmatrix} = -5 \begin{vmatrix} a_p & a_q \\ b_p & b_q \end{vmatrix}$$

§ 11.
$$\sum_{n=1}^{n} a_{nk} = \frac{\{ a_k - a_{(n+1)k} \} + (-1)^{k+1} \{ a_0 - a_{nk} \}}{1 + (-1)^k - M_k}.$$

§§ 12-13. $M_p$ is prime to the preceding terms of the series if $p$ is either prime or of the form $2^n$.

§ 14. Some relations between $N_r$ and $M_s$.

# ON DIFFERENTIAL EQUATIONS WITH THE GROUP PROPERTY

D. D. KOSAMBI,

*Muslim University, Aligarh.*

The purpose of the present work is the discussion of the possible types of homogeneous linear differential equations of which the solutions generate continuous groups of transformations in a space of $n$ dimensions. The geometrical interpretations of the results form perhaps the sole interesting element in a tedious formal presentation. A more elegant form of reasoning is greatly to be wished for, but seems to be impossible at present. The entire paper considers problems inverse to the usual discussions of continuous groups of deformation of a space into itself.

**1.** A given set of $r$ vectors of $n$ components each is said to generate a continuous group if the alternant of any two $\lambda$, $\mu$,

$$(\lambda^r \mu^i{}_{,r} - \mu^r \lambda^i{}_{,r}) = [\lambda, \mu] \qquad \ldots \quad (1.1)$$

is also a vector of the set, or more generally, a vector of the $r$ dimensional manifold defined by the vector fields in a space of $n$ dimensions. The repeated index denotes summation over all $n$ values and the subscript with a comma is used for ordinary partial differentiation. The actual transformations are derived from the infinitesimal generators

$$\lambda^i \frac{\partial}{\partial x^i}, \; \mu^i \frac{\partial}{\partial x^i} \ldots,$$

and it is seen that our definition of the group condition is essentially the same as that usually given.

Consider first the case where the set satisfies a system of the first order of partial differential equations :

$$\text{(A)} \qquad u^i{}_{,j} = \phi^i_j (x, u)$$

there being $r$ and only $r$ solutions $u = \lambda, \mu, \ldots \ldots$, etc.

The group condition can be worked out at once from

$$[\lambda, \mu]^i{}_{,j} = \phi^i_j(x, [\lambda, \mu]) \qquad \ldots \quad (1.2)$$

which reduces to

$$\phi^r{}_j(x, \lambda)\, \phi_r{}^i(x, \mu) - \phi_j{}^r(x, \mu)\, \phi^i{}_r(x, \lambda)$$
$$+ \lambda^r \phi^i{}_{r,j}(x, \mu) - \mu^r \phi^i{}_{r,j}(x, \lambda)$$
$$+ \frac{\lambda^r \partial \phi^i{}_r}{\partial \mu^k} \phi^k{}_j(x, \mu) - \mu^r \frac{\partial \phi^i{}_r}{\partial \lambda^k} \phi_j{}^k(x, \lambda)$$
$$- \phi^i{}_j(x, \lambda^r \phi_r{}^i(x, \mu) - \mu^r \phi^i{}_r(x, \lambda)) = 0. \qquad \ldots \quad (1.3)$$

With these, however, there must be taken the compatibility conditions of (A), of the form

$$u^i{}_{j,k} - u^i{}_{k,j} \equiv \phi^i{}_{j,k}(x, u) - \phi^i{}_{k,j}(x, u) + \frac{\partial \phi^i{}_j}{\partial u^r} \phi^r{}_k - \frac{\partial \phi^i{}_k}{\partial u^r} \phi_j{}^r = 0$$

$$i.e. \qquad \psi^i{}_{jk}(x, u) = 0. \qquad \ldots \quad (1.4)$$

These must hold identically, or in virtue of (A), or must themselves be taken as the equations of an extended system to be adjoined to A.

If we form the antisymmetric product

$$\lambda^k \psi^i{}_{jk}(\mu) - \mu^k \psi^i{}_{jk}(\lambda)$$

we have

$$\lambda^k \phi^i{}_{j,k}(x, \mu) - \mu^k \phi_{j,k}(x, \lambda) - \lambda^k \phi^i{}_{k,j}(x, \mu) + \mu^k \phi^i{}_{kj}(x, \lambda)$$
$$+ \lambda^k \phi^r{}_k(x, \mu) \frac{\partial \phi^i{}_j}{\partial \mu^r} = 0. \qquad \ldots \quad (1.5)$$

Of this, four terms coincide with those in (1.3), but for sign. Adding the two, we obtain a further system which, when taken with the equations of compatibility, is to replace the group condition :

$$\phi^r{}_j(x, \lambda)\, \phi^i{}_r(x, \mu) - \phi^r{}_j(x, \mu)\, \phi^i{}_r(x, \lambda)$$
$$- \phi^i{}_j(x, \lambda^r \phi^i{}_r(x, \mu) - \mu^r \phi^i{}_r(x\lambda))$$
$$+ \lambda^r \phi^i{}_{j,r}(x, \mu) - \mu^r \phi_{j,r}(x, \lambda)$$
$$+ \lambda^r \phi_r{}^h(x, \mu) \frac{\partial \phi^i{}_j}{\partial \mu^h} - \mu^h \phi^r{}_h(x, \lambda) \frac{\partial \phi^i}{\partial x^r} = 0$$
$$\ldots \quad (1.6)$$

**2.** Since the formulae of the last section are utterly unwieldy, and incapable of being interpreted to any particular extent, I shall consider a special case. We recall the condition that $a\lambda + b\mu$ is a generator with $\lambda$ and $\mu$, $a$ and $b$ being arbitrary constants not both zero. To this end, it is sufficient that the system (A) be homogeneous. We take then, the system

$$\text{(B)} \qquad u^i_{\ |j} \equiv u^i_{\ ,j} + \Gamma^i_{jk}\, u^k = 0.$$

This is to be regarded as tensor invariant and it then follows at once that the $\Gamma^i_{jk}$ can be taken to form the co-efficients of affine connections for an $n$ dimensional manifold. The group and compatibility conditions are respectively:

$$\Omega^h_{\ ij|k}\, \lambda^i \mu^j = 0\,; \qquad \Omega^h_{\ ij} = \Gamma^h_{ij} - \Gamma^h_{ji} \qquad \dots \quad (2.1)$$

$$u^l\, R^h_{\ ljk} = 0. \qquad \dots \quad (2.2)$$

Here, the tensor R is the Riemann-Christoffel tensor of the connection, and $\Omega$ is twice the torsion-tensor. The vertical bar is used for covariant differentiation and the equations (B) are simply those of a parallel vector. The case where (2.2) are identically satisfied is trivial, since the space then becomes Euclidean, the R's vanishing identically and with proper choice of co-ordinates the $\Gamma$'s also. In case the equations (2.2) are not identically satisfied, further equations may be derived by covariant differentiation. Of all these, the distinct ones must be finite in number, say $q$, that have precisely the $r$ independent solutions that form our generators. From each of these, a certain number of bilinear forms in $\lambda$, $\mu$ can be built up. Now if the group condition is not to place extra restrictions on the solutions, then the form (2.1) must belong to the field of the forms derivable from our compatibility conditions. We may sum up the results by:

*The homogeneous system (B) possesses the group property if and only if (a) the torsion-tensor has a vanishing covariant derivative, or (b) the covariant derivative of the torsion-tensor can be linearly expressed in terms of the curvature tensor $R^i_{\ jkl}$ and its first $q-1$ covariant derivatives.*

As a corollary, it is seen that the symmetric affine connections, possess the group property for parallel fields. In all cases, it must be remembered that the connection is to admit $r$ and only $r$ independent parallel fields.

**3.** The case is different for equations of the higher order. We consider homogeneous equations of the second order, and again postulate a first order operator of covariant differentiation. In particular, take the system

$$u^i_{|jk} = A^i_{jkl}\, u^l \tag{C}$$

This has the compatibility conditions :

$$[A^h_{ijk} - A^h_{jik} R^h_{kij}]\, u^k = 0. \tag{3.1}$$

$$\begin{aligned} u_{|hr} R^r_{ijk} - u^l_{|i} R^h_{ljk} - u^l_{|k} A^h_{ijl} \\ + u^l_{|j} A^h_{ikl} + u^l\, [A^h_{ikl\|j} - A^h_{ijl\|k}] = 0 \end{aligned} \tag{3.2}$$

Using now the fact that $\lambda$, $\mu$ are both solutions of (C), we proceed to compute the second covariant derivative of the alternant, and find the group property to be

$$\begin{aligned} (\lambda^l_{|i}\, \mu^k - \mu^l_{|i}\, \lambda^k)\, A^h_{ljk} + (\lambda^r \mu^h_{|l} - \mu^r \lambda^h_{|l})\, A^l_{ijr} \\ -(\lambda^r \mu^l_{|r} - \mu^r \lambda^l_{|r})\, A^h_{ijl} \\ + (\lambda^l_{|j}\, \mu^r - \mu^l_{|j}\, \lambda^r)\, [A^h_{lir} - A^h_{ril}] \\ + (\lambda^l \mu^r\, [A^h_{lir|j} - A^h_{ril|j}] = 0 \end{aligned} \qquad \ldots \quad (3.3)$$

From (3.2), we can derive as before a binary expression in $\lambda$, $\mu$. This, however, is no longer a form. Eliminating the curvature tensor by means of (3.1) we find an equation of the same type as (3.3). The two coincide when and only when the tensor $A^i_{jkl}$ obeys the same subscript identities as the curvature tensor, and actually coincides with it, or when the tensor $A^i_{jkl}$ vanishes identically. In the former case we find the equations to be those of "*affine collineation.*"

A similar result seems to hold for those homogeneous equations wherein first covariant derivatives are assumed to enter : for any symmetric connection, the first covariant derivative cannot enter without extra conditions on the solutions, and a consequent loss of the group property. As the computations are too long to trust without verification, I omit them, and state the result in the form of a conjecture.

*For symmetric affine connections, the only homogeneous partial differential equations of the second order possessing the group property are*

$$u^i_{\mid jk} = 0 \qquad \text{and} \qquad u^i_{\mid jk} = R^i_{\ jkl}\, u^l$$

**4.** There are still some considerations which remain. For instance, it may be possible for a certain subset of the set of solutions to possess the group property. This would mean essentially that the group conditions do introduce new restrictions on the solutions, in the form of further partial differential equations which form a completely integrable system with the original one, but naturally with fewer common solutions. There is again the line of approach indicated by the classical theorems of Lie on the constants of compositions of a group; geometrically, this leads to the investigation of the Ricci "coefficients of rotation," and is of special interest for transitive groups, since here it is possible to associate certain covariant vectors with the contravariant vectors that form the solutions under consideration. A case of some interest is also furnished by the introduction of a metric in the space, which allows among other things, the equations of Killing.

The second order case is of interest because of the highly specialized type of possible systems. The principal system corresponds to the "equations of variation" for the paths of the space, as follows from my work on parallelism. The same paper (under publication) shows that the general covariant derivative is non-distributive, and may lead to non-homogeneous linear equations, a case not considered here since the methods outlined give a successful treatment of that type as well.

# ON SOME FORMULAE IN THE THEORY OF THE ZETA-FUNCTION*

V. L. MUTATKER,

*Research Scholar, Allahabad University.*

In the present paper, I seek to express $\zeta(s) \prod_{p/k} \left(1 - \frac{1}{p^s}\right)$, (where $p/k$ means that $p$ runs through all the prime divisors of $k$), in the form of definite integrals and contour integrals of the Hankel type.

**1.** From the definition of $\zeta(s, a)$ we have

$$\zeta(s, a/k) = \sum_{n=0}^{\infty} \frac{1}{(a/k + n)^s} = k^s \sum_{n=0}^{\infty} \frac{1}{(a + kn)^s} \quad \ldots \quad (1.01)$$

Also,

$$1/\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right) \dagger$$

Hence

$$\zeta(s) \prod_{p/k} \left(1 - \frac{1}{p^s}\right) = \sum_{(n,\, k)=1} \frac{1}{n^s}$$

$$= \sum_{\substack{a<k \\ (a,\, k)=1}} \sum_{n=0}^{\infty} \frac{1}{(a + kn)^s}$$

$$= \frac{1}{k^s} \sum_{\substack{a<k \\ (a,\, k)=1}} \zeta(s, a/k) \text{ by } (1.01) \quad \ldots \quad (1.02)$$

where $p/k$ has the same meaning as before and $(a, k) = 1$ means that $a$ and $k$ are relatively prime.

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* I wish to express my indebtedness to Dr. P. L. Srivastava, D. Phil., for his kind interest and assistance.

† Whittaker and Watson, *Modern Analysis*, 1920 Ed., p. 272.

**1.1.** ***A Definite integral for*** $\zeta(s) \prod_{p/k} \left(1 - \frac{1}{p^s}\right)$.

Now since*

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax}}{1 - e^{-x}}\, dx$$

we obtain from (1.02)

$$\zeta(s) \prod_{p/k} \left(1 - \frac{1}{p^s}\right) = \frac{1}{k^s} \sum \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax/k}}{1 - e^{-x}}\, dx$$

$$= \frac{1}{k^s} \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} \sum e^{-ax/k}}{1 - e^{-x}}\, dx \qquad \text{... (1.1)}$$

the summations being extended to all values of $a < k$ for which $(a, k) = 1$.

**1.2.** ***A Contour integral for*** $\zeta(s) \prod_{p/k} \left(1 - \frac{1}{p}\right)$.

From Hankel's expression† for $\zeta(s, a)$ as a contour integral, we obtain

$$\zeta(s, a/k) = -\frac{\Gamma(1-s)}{2\pi i} \int_\infty^{(0+)} \frac{(-z)^{s-1} e^{-az/k}}{1 - e^{-z}}\, dz.$$

Substituting this value in (1.02), we have

$$\zeta(s) \prod_{p/k} \left(1 - \frac{1}{p^s}\right) = \frac{1}{k^s} \sum -\frac{\Gamma(1-s)}{2\pi i} \int_\infty^{(0+)} \frac{(-z)^{s-1} e^{-az/k}}{1 - e^{-z}}\, dz$$

$$= -\frac{\Gamma(1-s)}{2\pi i k^s} \int_\infty^{(0+)} \frac{(-z)^{s-1} \sum e^{-az/k}}{1 - e^{-z}}\, dz \qquad \text{... (1.2)}$$

the summations extending to all $a$ for which $a < k$, $(a, k) = 1$.

**1.3.** ***Another expression for***

$$\zeta(s) \prod_{p/k} \left(1 - \frac{1}{p^s}\right).$$

With the origin as centre and radius $\rho = n + \frac{1}{2}$, describe the semicircle LNM and a small semi-circle of radius $\delta$.

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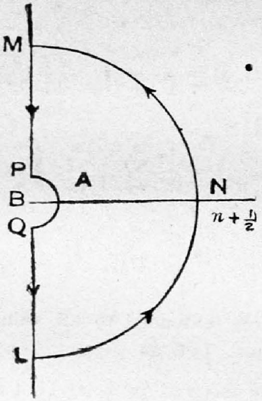
* Whittaker and Watson, p. 266.

† Whittaker and Watson, p 266.

Now integrating the function $(kz + a)^{-s} / (e^{2\pi i z} - 1)$ over the




closed contour S, *i.e.*, QLNMPAQ, we have by Cauchy's theorem

$$\sum_{1}^{n} (kn + a)^{-s} = \int_{S} \frac{(kz + a)^{-s}}{e^{2\pi i z} - 1}\, dz$$

$$= \int_{(AQ)} + \int_{(QL)} + \int_{(LN)} + \int_{(NMPA)}$$

The last integral on the right-side is equivalent to

$$-\int_{(NMPA)} (kz + a)^{-s}\, dz - \int_{(NMPA)} \frac{(kz + a)^{-s}}{e^{-2\pi i z} - 1}\, dz$$

$$= \int_{(AN)} (kz + a)^{-s}\, dz - \int_{(NMPA)} \frac{(kz + a)^{-s}}{e^{-2\pi i z} - 1}\, dz.$$

We therefore have

$$\sum_{1}^{n} (kn + a)^{-s} = \int_{(AN)} (kz + a)^{-s}\, dz + \int_{(QL)} \frac{(kz + a)^{-s}}{e^{2\pi i z} - 1}\, dz$$

$$+ \int_{(PM)} \frac{(kz + a)^{-s}}{e^{-2\pi i z} - 1}\, dz + \int_{(AQ)} \frac{(kz + a)^{-s}}{e^{2\pi i z} - 1}$$

$$- \int_{(PA)} \frac{(kz + a)^{-s}}{e^{-2\pi i z} - 1}\, dz + \int_{(LN)} \frac{(kz + a)^{-s}}{e^{2\pi i z} - 1}$$

$$- \int_{(NM)} \frac{(kz + a)^{-s}}{e^{-2\pi i z} - 1}\, dz$$

$$= \int_{(AN)} (kz + a)^{-s}\, dz$$

$$+ 2 \int_{\delta}^{(n+\frac{1}{2})} (a^2 + k^2y^2)^{-s/2} \left[ \sin s \tan^{-1} \frac{ky}{a} \right] \frac{dy}{e^{2\pi y} - 1}$$

$$+ \int_{(AQ)} \frac{(kz + a)^{-s}}{e^{2\pi iz} - 1}\, dz - \int_{(PA)} \frac{(kz + a)^{-s}}{e^{-2\pi iz} - 1}\, dz$$

$$+ \int_{(LN)} \frac{(kz + a)^{-s}}{e^{2\pi iz} - 1}\, dz - \int_{(NM)} \frac{(kz + a)^{-s}}{e^{-2\pi iz} - 1}\, dz \ldots \quad (1.31)$$

Now we proceed to show that each of the last two integrals on the right-side of (1.31) tends to 0 as $\rho$ (and hence $n$) becomes infinite. Over the arc LN

$$\left| \frac{z\,(kz + a)^{-s}}{e^{2\pi iz} - 1} \right| = O\,(\rho^{1-\sigma})$$

which tends to zero as $\rho \to \infty$ provided $\sigma$, the real part of $s$, is greater than unity.

Similarly, over the arc NM

$$\left| \frac{z\,(kz + a)^{-s}}{e^{-2\pi iz} - 1} \right| = O\,(\rho^{1-\sigma})$$

which tends to zero as $\rho \to \infty$ if $\sigma > 1$.

Again

$$\frac{z\,(kz + a)^{-s}}{e^{-2\pi iz} - 1} \to -\frac{a^{-s}}{2\pi i}$$

as $\delta \to 0$. Therefore

$$\lim_{\delta \to 0} \int_{(PA)} \frac{(kz + a)^{-s}}{e^{-2\pi iz} - 1}\, dz = \frac{a^{-s}}{4}$$

Similarly

$$\lim_{\delta \to 0} \int_{(AQ)} \frac{(kz + a)^{-s}}{e^{2\pi iz} - 1}\, dz = -\frac{a^{-s}}{4}.$$

From (1.31), we therefore have, if $\sigma > 1$,

$$\sum^{\infty} (kn+a)^{-s} = -\frac{a^{-s}}{2} + \int_0^{\infty} (kx+a)^{-s}\, dx$$

$$+ 2\int_0^{\infty} (a^2+k^2y^2)^{-s/2} \left[\sin s \tan^{-1}\frac{ky}{a}\right] \frac{dy}{e^{2\pi y}-1}$$

$$= -\frac{a^{-s}}{2} + \frac{k^{-1}a^{1-s}}{s-1}$$

$$+ 2\int_0^{\infty} (a^2+k^2y^2)^{-s/2} \left[\sin s \tan^{-1}\frac{ky}{a}\right] \frac{dy}{e^{2\pi y}-1}$$

or

$$\sum_0^{\infty} (kn+a)^{-s} = \frac{a^{-s}}{2} + \frac{k^{-1}a^{1-s}}{s-1}$$

$$+ 2\int_0^{\infty} (a^2+k^2y^2)^{-s/2} \left[\,[\sin s \tan^{-1}\frac{ky}{a}\right] \frac{dy}{e^{2\pi y}-1} \qquad \dots (1.32)$$

It is easy to see that the integral involved converges for all values of $s$. Hence it defines an analytic function for all values of $s$.

Therefore the result of (1.32) taken with the formula (1.02) gives

$$\zeta(s) \prod_{p/k}\left(1-\frac{1}{p^s}\right) = \frac{1}{2}\sum a^{-s} + \frac{k^{-1}}{s-1}\sum a^{1-s}$$

$$+ 2\sum \int_0^{\infty} (a^2+k^2y^2)^{-s/2} \left[\sin s \tan^{-1}\frac{ky}{a}\right] \frac{dy}{e^{2\pi y}-1} \qquad \dots (1.33)$$

the summations being extended to all values of $a < k$ for which $(a, k) = 1$.

The above formula can also be obtained directly by making use of Hermite's expression* for $\zeta(s, a)$.

* Whittaker and Watson, pp. 269-270.

# ON CENTRAL OVALOIDS.

BY P. GANAPATI,

*Research Student, Annamalai University.**

**1.** An ovaloid is a closed convex surface, and may be defined, according to Minkowski†, as a closed point-set which has in common with an arbitrary straight line in space a finite strip of the line, or a single point, or no point at all, and does not lie completely in a plane. For the purposes of this paper, we shall assume that the principal radii of curvature at any point of the ovaloid exist and are both finite.

The main result of this paper is a generalisation of a theorem on ovals‡ given by me, namely, that the necessary and sufficient condition that an oval has a centre is that the curvatures are equal at opposite points of it.

**2.** Let $\rho_u$, $\rho_v$ be the principal radii of curvature at a point P where the lines of curvature are the parametric curves $u$ = constant, and $v$ = constant; $\rho_u'$, $\rho_v'$ the corresponding principal radii of curvature at the *opposite* point P′ (where the tangent plane is parallel to the tangent plane at P), the principal lines of curvature being $u'$ = constant, and $v'$ = constant. The surface formed by the mid. points of diameters of the ovaloid will be called the *medial surface*.

THEOREM I. *If the principal directions at every pair of opposite points of an ovaloid are parallel, the principal radii of curvature at a corresponding point of the medial surface are given by*

$$\rho_u^* = \tfrac{1}{2}(\rho_u - \rho'_u),$$

*and*

$$\rho_v^* = \tfrac{1}{2}(\rho_v - \rho'_v).$$

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* Published with the permission of the University authorities.

† Minkowski : *Gesammelte Abhandlungen* II, p. 131.

‡ *Annamalai University Journal*, Vol. I, No. 1, Theorems on Ovals (Th. 2 and 3).

If $\mathbf{R}$ is the vector to a point P of the ovaloid, the principal magnitudes of the first and second order at the point P are given by the following formulae*:

$$\left.\begin{array}{lll} E = \mathbf{R}_1{}^2 \neq 0, & F = \mathbf{R}_1 . \mathbf{R}_2 = 0, & G = \mathbf{R}_2{}^2 \neq 0, \\ L = \mathbf{R}_{11} . \mathbf{N} \neq 0, & M = \mathbf{R}_{12} . \mathbf{N} = 0, & N = \mathbf{R}_{22} . \mathbf{N} \neq 0; \end{array}\right\} \quad \ldots \quad (1)$$

$$\left.\begin{array}{ll} k_u = \dfrac{1}{\rho_u} = \dfrac{L}{E}, & k_v = \dfrac{1}{\rho_v} = \dfrac{N}{G}, \\ \mathbf{N}_1 = -k_u \mathbf{R}_1, & \mathbf{N}_2 = -k_v \mathbf{R}_2; \end{array}\right\} \quad \ldots \quad (2)$$

$$\left.\begin{array}{l} \mathbf{R}_{11} = \dfrac{E_1}{2E} \mathbf{R}_1 - \dfrac{E_2}{2G} \mathbf{R}_2 + L\mathbf{N} \\ \mathbf{R}_1{}^2 = \dfrac{E_2}{2E} \mathbf{R}_1 + \dfrac{G_1}{2G} \mathbf{R}_2 \\ \mathbf{R}_{22} = -\dfrac{G_1}{2E} \mathbf{R}_1 + \dfrac{G_2}{2G} \mathbf{R}_2 + N\mathbf{N}. \end{array}\right\} \quad \ldots \quad (3)$$

In the above formulae, the suffix 1 denotes differentiation with respect to $u$ and the suffix 2 differentiation with respect to $v$.

Let $\mathbf{R}$ and $\mathbf{R}'$ be the position vectors of a pair of opposite points P, P′ of the ovaloid, so that

$$\mathbf{R}' = \mathbf{R} + x\, \mathbf{R}_1 + y\, \mathbf{R}_2 + z\, \mathbf{N}. \quad \ldots \quad (4)$$

Now

$$\mathbf{R}_1' \frac{\partial u'}{\partial u} + \mathbf{R}'_2 \frac{\partial v'}{\partial u} = \frac{\partial \mathbf{R}'}{\partial u} \quad \ldots \quad (5)$$

$$= \left(1 + x_1 + \frac{E_1 x + E_2 y}{2E} - k_u z\right) \mathbf{R}_1$$

$$+ \left(\frac{-E_2 x + G_1 y}{2G} + y_1\right) \mathbf{R}_2 + (Lx + z_1)\, \mathbf{N}, \quad \ldots \quad (5')$$

by differentiating (4), and using (2) and (3).

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* The Notation is that of Weatherburn : *Differential Geometry of three dimensions, I.*

*Vide* also Mellish : *Annals of Mathematics*, Vol. 32, No. 1 (1931) : Notes on Differential Geometry, pp. 184—189.

Similarly,

$$R'_1 \frac{\partial u'}{\partial v} + R'_2 \frac{\partial v'}{\partial v} = \frac{\partial R'}{\partial v} \quad \dots \quad (6)$$

$$= \left(\frac{E_2 x - G_1 y}{2E} + x_2\right) R_1$$

$$+ (1 + y_2 + \frac{G_1 x + G_2 y}{2G} - k_v z) R_2 + (Ny + z_2) N, \quad (6')$$

by differentiation.

Here

$$R_1' = \frac{\partial R'}{\partial u'} \text{ and } R_2' = \frac{\partial R'}{\partial v'}.$$

Also since $R_1'$ and $R_2'$ are parallel to $R_1$ and $R_2$ respectively, it follows* that

$$\frac{\partial v'}{\partial u} = 0, \frac{\partial u'}{\partial v} = 0.$$

$$\left.\begin{aligned} &\text{Therefore,} \quad \frac{-E_2 x + G_1 y}{2G} + y_1 = 0, \quad Lx + z_1 = 0; \\ &\text{and} \quad \frac{E_2 x - G_1 y}{2E} + x_2 = 0, \quad Ny + z_2 = 0, \end{aligned}\right\} \quad \dots \quad (7)$$

---

* If **N** be the unit normal at P, we take the unit normal at the opposite point P′ ($u'$, $v'$) such that

$$N' = -N. \quad \dots \quad (A)$$

If the parametric curves $u'$=const., $v'$ = const. at P′ be the lines of curvature, then from (A), we have

$$\frac{\partial N'}{\partial u'} = -\frac{\partial N}{\partial u'} = -\frac{\partial N}{\partial u}\frac{\partial u}{\partial u'} - \frac{\partial N}{\partial v}\frac{\partial v}{\partial u'}.$$

Hence from (2) and the corresponding relations at P′,

$$-k'_u R'_1 = + k_u R_1 \frac{\partial u}{\partial u'} + k_u R_2 \frac{\partial v}{\partial u'}.$$

But by hypothesis, $R'_1$ and $R'_2$ the principal directions are parallel to $R_1$ and $R_2$.

$$\therefore \quad \frac{\partial v}{\partial u'} = 0, \text{ and similarly } \frac{\partial u}{\partial v'} = 0.$$

The equations (5′) and (6′) thus reduce to

$$R'_1 \frac{\partial u'}{\partial u} = \left(1 + x_1 + \frac{E_1 x + E_2 y}{2E} - k_u z\right) R_1,$$

and

$$R'_2 \frac{\partial v'}{\partial v} = 1 + y_2 + \frac{G_1 x + G_2 y}{2G} - k_v z) R_2.$$

If the parallel vectors $R_1'$ and $R_1$ are measured positively in opposite directions,

$$\frac{1}{\sqrt{E'}} R'_1 = -\frac{1}{\sqrt{E}} R_1, \text{ being unit parallel vectors,}$$

and

$$\frac{1}{\sqrt{G'}} R'_2 = -\frac{1}{\sqrt{G}} R_2.$$

We have, therefore,

$$\left.\begin{aligned} -\sqrt{\frac{E'}{E}} \frac{\partial u'}{\partial u} &= 1 + x_1 + \frac{E_1 x + E_2 y}{2E} - k_u z \\ -\sqrt{\frac{G'}{G}} \frac{\partial v'}{\partial v} &= 1 + y_2 + \frac{G_1 x + G_2 y}{2G} - k_v z \end{aligned}\right\} \quad \ldots \quad (8)$$

We shall now investigate the corresponding quantities for the medial surface, which we shall always denote by the addition of a star (*).

The vector to the mid. point Q of the diameter PP′ is

$$R^* = \tfrac{1}{2}(R + R') = \tfrac{1}{2}(2R + xR_1 + yR_2 + zN) \qquad \text{from (4)}$$

Now

$$\frac{\partial R^*}{\partial u} = R_1{}^* \frac{\partial u^*}{\partial u} + R_2{}^* \frac{\partial v^*}{\partial u},$$

where the reference curves $u^* = \text{const.}$, and $v^* = \text{const.}$ are the lines of curvature on the medial surface and $R_1{}^* = \frac{\partial R^*}{\partial u^*}$, $R_2{}^* = \frac{\partial R^*}{\partial v^*}$,

$$\therefore \qquad \frac{\partial R^*}{\partial u} = \tfrac{1}{2} R_1\left(2 + x_1 + \frac{E_1 x + E_2 y}{2E} - k_u z\right)$$

$$+ \tfrac{1}{2} R_2 \left( y_1 + \frac{-E_2 x + G_1 y}{2G} \right) + \tfrac{1}{2} N (Lx + z_1)$$

by differentiation,

$$= \tfrac{1}{2} R_1 \left( 1 - \sqrt{\frac{E'}{E}} \; \frac{\partial u'}{\partial u} \right) \text{ by virtue of (7) and (8).}$$

$$\dots \quad (9')$$

Similarly,

$$R_1{}^* \frac{\partial u^*}{\partial v} + R_2{}^* \frac{\partial v^*}{\partial v} = \frac{\partial R^*}{\partial v}$$

$$= \tfrac{1}{2} R_2 \left( 1 - \sqrt{\frac{G'}{G}} \; \frac{\partial v'}{\partial v} \right) \quad \dots \quad (9'')$$

This shows that the tangent plane to the medial surface at Q is parallel to the tangent planes at P and P′. Hence we can take unit normal at Q in the same sense as that at P and write

$$N^* = N,$$

so that

$$\frac{\partial N^*}{\partial u} = N_1{}^* \frac{\partial u^*}{\partial u} + N_2{}^* \frac{\partial v^*}{\partial u}$$

$$= - k_u{}^* R_1{}^* \frac{\partial u^*}{\partial u} - k_u{}^* R_2{}^* \frac{\partial v^*}{\partial u},$$

where $N_1{}^*$, $N_2{}^*$ stand for $\frac{\partial N^*}{\partial u^*}$, $\frac{\partial N^*}{\partial u^*}$ respectively, and $k_u{}^*$, $k_v{}^*$ are the principal curvatures of the medial surface at Q.

But
$$\frac{\partial N^*}{\partial u} = \frac{\partial N}{\partial u} = - k_u R_1.$$

$$\therefore \qquad k_u R_1 = k_u{}^* R_1{}^* \frac{\partial u^*}{\partial u} + k_v{}^* R_2{}^* \frac{\partial v^*}{\partial u},$$

and from (9′),

$$\tfrac{1}{2} R_1 \left( 1 - \sqrt{\frac{E'}{E}} \; \frac{\partial u'}{\partial u} \right) = R_1{}^* \frac{\partial u^*}{\partial u} + R_2{}^* \frac{\partial v^*}{\partial u}.$$

Now we can always take $\frac{\partial u^*}{\partial u} \neq 0$. If also $\frac{\partial v^*}{\partial u} \neq 0$, we have, on

comparing these equations,

$$k_u^* = k_v^* = \frac{2\,k_u}{1 - \sqrt{\frac{E'}{E}}\,\frac{\partial u'}{\partial u}},$$

which shows that Q is an umbilic of the medial surface. Hence unless the medial surface is a sphere, we must have

$$\frac{\partial v^*}{\partial u} = 0.$$

Similarly $\frac{\partial u^*}{\partial v} = 0$, showing that the vectors $\mathbf{R}_1^*$, $\mathbf{R}_2^*$ are respectively parallel to $\mathbf{R}_1$ and $\mathbf{R}_2$. Thus

*the principal directions at a point Q of the medial surface are parallel to the corresponding principal directions at the ends of the diameter PP′ which is bisected at Q.*

Now

$$\frac{1}{\sqrt{E^*}}\,\mathbf{R}_1^* = \frac{1}{\sqrt{E}}\,\mathbf{R}_1,$$

and

$$\frac{1}{\sqrt{G^*}}\,\mathbf{R}_2^* = \frac{1}{\sqrt{G}}\,\mathbf{R}_2.$$

∴ from (9′) and (9′′),

$$\left.\begin{aligned} \frac{\partial u^*}{\partial u}\sqrt{\frac{E^*}{E}} &= \tfrac{1}{2}\left(1-\sqrt{\frac{E'}{E}}\,\frac{\partial u'}{\partial u}\right) \\ \frac{\partial v^*}{\partial v}\sqrt{\frac{G^*}{G}} &= \tfrac{1}{2}\left(1-\sqrt{\frac{G'}{G}}\,\frac{\partial v'}{\partial v}\right) \end{aligned}\right\} \quad \dots \quad (10)$$

If $\rho_u$, $\rho_v$ are the principal radii of curvature at P, these equations give

$$\frac{\rho_u^*}{\rho_u} = \tfrac{1}{2}\left(1-\frac{\rho'_u}{\rho_u}\right) \quad \text{and} \quad \frac{\rho_v^*}{\rho_v} = \tfrac{1}{2}\left(1-\frac{\rho'_v}{\rho_v}\right).$$

Or $$\rho_u^* = \tfrac{1}{2}(\rho_u - \rho_u'),\ \rho_v^* = \tfrac{1}{2}(\rho_v - \rho'_v), \quad \dots \quad (11)$$

which proves Theorem I.

**3.** As an immediate deduction from Theorem I, we have

THEOREM II. *The necessary and sufficient condition that an ovaloid has a centre is that the principal directions at pairs of opposite points of it should be parallel and the corresponding principal curvatures should be respectively equal.*

The condition is necessary. For if the ovaloid has a centre, the element of the surface in the neighbourhood of a point P is congruent to the element of the surface at the opposite point P′. Hence the principal directions at every pair of opposite points are parallel and the corresponding principal curvatures equal.

To prove that the condition is also sufficient, we have, from equation (9′) above

$$\frac{\partial R^*}{\partial u} = \tfrac{1}{2} R_1 \left(1 - \sqrt{\frac{E'}{E}}\ \frac{\partial u'}{\partial u}\right)$$

$$= \tfrac{1}{2} R_1 \left(1 - \frac{k_u}{k'_u}\right)$$

since the principal directions at opposite points are parallel,

and
$$\frac{\partial R^*}{\partial v} = \tfrac{1}{2} R_2 \left(1 - \frac{k_v}{k'_v}\right).$$

Hence if in addition $k_u = k'_u$ and $k_v = k'_v$,

$$\frac{\partial R^*}{\partial u} = 0, \quad \frac{\partial R^*}{\partial v} = 0,$$

showing that the vector to the mid. point of every diameter PP′ of the ovaloid is a constant vector, *i.e.*, the ovaloid has a centre.

By means of this theorem, we can deduce a theorem* of Blaschke and Hessenberg, *viz.*,

*If every orthogonal projection of an ovaloid has a centre, the ovaloid itself has a centre.*

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* W. Blaschke und G. Hessenberg: Lehrsätze über konvexe körper, *Jahresbericht d. D. Math. Ver.*, **26**.

Let a circumscribed cylinder of the ovaloid touch it at two opposite points P $(u, v)$, P$'$ $(u', v')$, and let $\alpha$ be the angle between the directions $v$ = const. and $v'$ = const. If now $\theta$ be the angle which the generators of the cylinder make with the direction $v$ = const. the radius of curvature R of the orthogonal section of the cylinder at $p$, the projection of P is given by†

$$R = \rho_u \cos^2 \theta + \rho_v \sin^2 \theta,$$

and the radius of curvature R$'$ of the same section at the opposite point $p'$ by

$$R' = \rho'_u \cos^2 (\theta + \alpha) + \rho'_v \sin^2 (\theta + \alpha).$$

Now since the projection has a centre, $R = R'$.

$$\therefore\ \rho_u \cos^2 \theta + \rho_v \sin^2 \theta = \rho'_u \cos^2 (\theta + \alpha) + \rho'_v \sin^2 (\theta + \alpha) \ldots \quad (12)$$

Now this equation is true for all values of $\theta$ ($\alpha$ remaining unchanged), for we can take circumscribed cylinders in various directions touching the ovaloid at P, P$'$, and in every case the section has a centre by hypothesis.

Hence in equation (12), the maximum and minimum values of the left-hand member are respectively equal to the maximum and minimum values of the right-hand member

$$\therefore\ \rho_u = \rho_u'$$

and

$$\rho_u = \rho_v'.$$

Substituting these values in (12), we have $\alpha = 0$ or a multiple of $\pi$, showing that the principal directions at P and P$'$ are parallel and the principal curvatures equal. Since we have taken P, P$'$ to be any pair of opposite points, we find from Theorem II that the ovaloid has a centre.

In conclusion I wish to express my indebtedness to Mr. A. Narasinga Rao for his kind help and advice in the preparation of this paper.

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† Blaschke, *Kreis und Kugel*, p. 117. Also Mellish, *loc. cit.*

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# NOTES AND QUESTIONS.

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AND

N. SUNDARAM IYER, M.A.

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# Notes and Questions.

## Recent Discoveries in the History of Mathematics.

It is well-known that the term algebra has been derived from the title of a work written in about 825 by a Persian mathematician named Mohammed ibn Musa al-Khowarizmi. The title of this work has been transliterated somewhat differently in various of the larger dictionaries as well as in the histories of mathematics. One of these is as follows: *al-jabr w'al muquabalah.* A common translation of the former of these terms, which actually gave rise to the word 'algebra,' is the transformation of an equation which involves positive and negative terms so that only positive terms appear in each of its members. Recently Professor O. Neugebauer supported the view expressed earlier by S. Ganz, that the second term of the given title is simply the Arabic translation of the former, which is of Assyrian origin and means an equation.

If this view is correct practically all of the explanations of the meaning of the term 'algebra' which appear in the literature are in need of revision. What is perhaps more interesting in this connection is the fact that there seems to be no scientific reason for adopting the common explanation of the meaning of this term. Professor O. Neugebauer expressed the view noted above in the opening note of Volume 2 (1932), *Quellen und Studien zur Geschichte der Mathematik*, part entitled "Studien." The first volume of the part entitled "Quellen" of this periodical is devoted to the Moscow Mathematical Papyrus, which was translated into German by W. W. Struve and published in 1930.

It is stated on page 169 of this volume that the ancient Egyptians used the rule that the area of a hemisphere is equal to two great circles of the corresponding sphere. The announcement that this very elegant rule appeared so early naturally attracted much attention and was widely published, but it is now generally believed that W. W. Struve failed to translate correctly in this instance and that it is probably not true that the ancient Egyptians knew this rule. On the other hand, the statements that

the ancient Egyptians found correctly the area of a general plane triangle and the volume of the frustum of a square pyramid, which also appear in this volume (pages 172 and 176 respectively) are now commonly accepted as established and they constitute important recent additions to our knowledge of ancient mathematics. It is especially interesting to know that the ancient Egyptians used a correct rule for finding the area of a triangle.

Recent discoveries relating to Babylonian mathematics have also greatly affected the history of ancient mathematics and have enabled us to put it on a much higher plane than was possible heretofore. In particular, all earlier accounts relating to the historical development of the quadratic equation have become antiquated during the last three or four years as a result of discoveries relating to the mathematical attainments of the Babylonians along algebraic lines during a period of at least 1500 years beginning about 2000 B.C. This period overlaps that of the early Greek mathematical activity and hence it establishes a continuity in algebraic developments which several years ago had not been known to exist. We now know at least 19 Babylonian problems which give rise to quadratic equations and with respect to 10 of these the details of the solutions are given. In some cases the method used corresponds to the modern method of completing the square and differs only from our modern procedure by omitting the double sign when the square root is extracted. Cf. ***Studien***, Volume I (1930), page 194, of the periodical noted above.

These recent discoveries relating to the early study of the quadratic equation not only greatly modify our view as regards the ancient Greek contributions along the same line but they tend to augment our appreciation of this equation as a useful element in human thinking. The facts that the need of solving this equation was recognized so early and that a complete solution thereof was found thousands of years later exhibit the fundamental importance of elementary algebra and the slowness of some of the mathematical developments. In this particular case progress was delayed by the fact that it depended upon an extension of the number concept so as to include negative and complex numbers. These extensions were not completed satisfactorily until about the beginning of the nineteenth century and hence such statements as the following : " The general quadratic as we know it to-day

was thus fully mastered by the Greek mathematicians" are apt to mislead the beginner in the history of mathematics.*

The early Babylonian work relating to the solution of the quadratic equation may throw some light on the fact that the ancient Greeks never found the two roots of such an equation even when both of these roots are positive. It seems that both the Babylonians and the Egyptians had an early period of comparatively great mathematical activity and then for many generations they accepted the mathematical results of their ancestors without adding thereto. One of the important elements of ancient Greek mathematics is the effort to advance beyond the results received from their predecessors, especially along the line of geometry. In algebra they exhibited a much more conservative spirit and seem to have been more like their Egyptian and Babylonian predecessors. In fact, as regards the concept of number they seem even to have taken a step backward as regards practical mathematics by advocating for a long time the view that unity and the positive rational fractions which are separately not equal to an integer should not be regarded as numbers. They also failed to extend the number concept so as to include therein the negative or the complex numbers and hence they could also not have solved the general quadratic geometrically. The history of the geometric construction of the complex roots of such an equation seems to have received too little attention thus far.†

The fact that the history of logarithms is vitally connected with two of the most fundamental simply isomorphic groups of mathematics, *viz.*, a special group of addition and a special group of multiplication, appears to me as one of the most important recent discoveries in the history of elementary mathematics. It is well known that J. Napier, J. Burgi, and others approached the subject of logarithms from the standpoint of two related infinite series. One of these is arithmetic and the other is geometric, and to the product of two terms of the latter, the sum of two terms of the former was made to correspond. A geometric series extended infinitely in both directions is a group even in the most restricted sense of this term if it is possible to find therein the

* D. E. Smith, *History of Mathematics*, Vol. 1 (1923), p. 126.

† Cf. G. A. Miller, *Mathematical Gazette*, Vol. 12 (1925), p. 500.

product of two of its terms, and such an arithmetic series is such a group if it is possible to find therein the sum of at least two of its terms.* When this condition is satisfied the former must involve unity and the latter must involve zero, and these are the identity elements of the group.

It therefore results that these early investigators of the steps leading to logarithms were dealing with two simply isomorphic groups and from this point of view it is very strange that some of them, including J. Napier and J. Burgi, did not make the identities of these groups correspond in their crude efforts to construct tables of logarithms. This is the more surprising in view of the fact that the identities of these groups had been made to correspond earlier by M. Stifel in his *Arithmetica integra*, 1544. These identities were also made to correspond in the later tables by H. Briggs which were constructed according to a plan devised by H. Briggs and J. Napier in consultation. This plan called not only for the correspondence of the identities of the groups formed by the two series in question but also for the correspondence of 10 in the geometric series and unity in the arithmetic series, and hence for a table which is equivalent to the modern table to the base 10.

Attention has frequently been called to the fact that a trace of logarithms appears in the equation $x^n \cdot x^m = x^{n+m}$, which is found implicitly in the *Elements* of Euclid and also in the works of Archimedes. The fact that such a trace appears also in positional arithmetic, and hence in the earliest known efforts along this line by the ancient Sumerians, seems not to have been explicitly noted until recently. In this positional arithmetic the identities of the two groups noted above are made to correspond and therefore we have in our common system of numerical notation not only a simple isomorphism between a multiplication group and an addition group but also an exhibition of the fundamental principles of logarithms. Hence the statement that "the invention of logarithms came on the world as a

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* G. A. Miller, *Proceedings of the National Academy of Sciences*, Vol. 17 (1931), p. 464.

bolt from the blue," which is quoted approvingly in Smith's *History of Mathematics*, Volume 2 (1925), page 514, fails to exhibit important historical contact and a uniting and clarifying principle of various early mathematical developments which can be studied effectively from this point of view. Most mathematical historians seek points of continuity and not of discontinuity in the development of our subject.

One of the striking recent discoveries in the history of elementary geometry is the fact that about one hundred and twenty-five years before the appearance of L. Mascheroni's *Geometria del compasso* (1797), a Danish mathematician, George Mohr, had treated the same subject in a small work entitled *Euclides Danicus* (1672) which was republished and translated into German in 1928. It is surprising that such an interesting result as the proof of the fact that the constructions which can be effected by means of a ruler and circle can also be effected by means of the circle alone failed to attract more attention in the latter part of the seventeenth century, especially since periodical publications had then already been inaugurated. When this fact was rediscovered towards the close of the eighteenth century it received wide attention, but the mathematical historians failed to become aware of the earlier discovery for about one hundred and thirty years notwithstanding their frequent reference to the result and the growing interest in the history of our subject.

It is to be hoped that future mathematical historians will not call L. Mascheroni a plagiarist for this oversight as has been done in the case of P. Guldin for a somewhat similar oversight as regards the theorem commonly known by his name. In this case the theorem had been forgotten for a much longer time since it appeared in the works of the Greek mathematician Pappus about 300 A.D. It should also be noted that the cases are not quite similar because it is known that P. Guldin and J. Kepler consulted the work of Pappus while L. Mascheroni is not known to have consulted the work of George Mohr. It is, however, not established that P. Guldin realised that the theorem now commonly known by his name actually appears in the work of Pappus as H. Wieleitner pointed out in one of his latest publications

entitled *Kepler-Festschrift*, 1930. Many now consult treatises without noticing all that is contained therein and if P. Guldin's contemporaries had known that this theorem appears in the works of Pappus it is likely that some of them would have called attention to it in the controversial literature of that day.

The preceding remarks relate to recent discoveries in regard to the history of the older developments in mathematics. The number of such discoveries relating to the more recent developments is very large and varied. I shall here refer to only one of them and shall select this from the field of finite groups because I happen to be more familiar with this field than with many other equally important ones. It is well-known that a finite set of group elements constitutes a group if it includes the product of every pair and the square of every one of them. The question naturally arises whether one of these conditions may imply the other, and what groups are generated by these elements when this implication exists. This question obviously presents itself at the very door of group theory and hence it would naturally be supposed that it would have been answered early in the development of this subject. On the contrary it seems to have received no attention until very recently.

The question is the more important since in some definitions of groups it seems to be implied that the former of these conditions always entails the latter. This is the case, for instance, as regards the definition of transformation groups which appears on page 99 of the second edition of H. Weyl's *Gruppentheorie und Quantenmechanik*, 1931, but it is not the case as regards the definition of abstract groups which appears two pages later. The six operators of order 4 of the quaternion group together with the identity clearly satisfy the former condition and yet they do not constitute a group. They are, however, the only non-commutative group elements which satisfy the former of the two given conditions without also satisfying the latter. It is evident that the $2^n - 1$ operators of order two in the abelian group of order $2^n$ and of type (1, 1, 1, ... ) also satisfy the former of these two conditions without satisfying the latter, and that the same is true of the set of

hree elements composed of any operator, its inverse and the identity, and of the set of two elements composed of any operator and the identity.

It has recently been proved that the four sets which have just been noted are the only possible ones in which the former of the two given conditions is satisfied but not the latter. (Cf. *Proceedings of the National Academy of Sciences*, Volume 18 (1932), page 100). On the contrary, the group of order 2 is obviously the only one which does not involve a set of generators such that the set includes the square of every operator contained therein without containing all the operators of the group. Hence it results that while the former of the two given conditions usually implies the latter, the latter implies the former in only one trivial case. It may be added that when a set of group elements which satisfies the condition that it includes the product of every pair of them is composed of more than three elements and involves an element whose order exceeds 4 then the set must always constitute a group.

According to the *Enciclopedia delle Matematiche Elementari*, Volume 1, part 2 (1932), page 20, the notion of group has received in the last fifty years a great extension as a result of including therein any operations or transformations, finite or infinite in number, performed on the entities of a given field, whenever the transformation resulting from any two of its transformations belongs itself to the set. This definition is obviously too comprehensive to enable us to develop an extensive theory therewith and in the preceding paragraphs we used the term group with its narrower commonly accepted meaning. It would probably be ineffective to protest against the use of the term group with the wider meaning just noted since the mathematical investigator should have considerable freedom even as regards the use of his terms, but the work of the historian is greatly increased by the tendency on the part of some modern writers to make extensive use of this latitude. The time has already arrived when one is in danger of being misunderstood if one uses the term group incidentally unless one states explicitly what meaning is expected to be assigned to this term.

G. A. MILLER.

## Lecture Notes.

## On the Foci of a Conicoid.

LEMMA (1). If a quadric cone C with vertex P has two common plane sections with a quadric S, then the line of intersection of the two planes of the sections must lie in the polar plane of P with respect to S.

For let U, V denote the two planes. Then $C = S + \lambda UV$. Since P is the vertex of C, the polar plane of P with respect to C, *i.e.*, with respect to $S + \lambda UV$, must vanish identically. But the polar plane of P with respect to $\lambda UV$ is of the form $\lambda_1 U + \lambda_2 V$. Hence if W is the polar plane of P with respect to S, we must have an identical relation of the form $W + \lambda_1 U + \lambda_2 V = 0$; *i.e.*, the line of intersection of U, V lies on W.

LEMMA (2). The cone C of Lemma (1) must have contact along two generators with the enveloping cone of S which has the vertex P.

For $C = S + \lambda UV$, and the enveloping cone C′ must be of the form $S + \lambda' W^2$. Hence $C - C' = \lambda UV - \lambda' W^2$ must be a cone with vertex P. Since the line of intersection of U and V does not, in general, pass through P, and since U, V, W have a common line, it follows $\lambda UV - \lambda' W^2$ can be a cone with vertex P only if it is the square of the plane W′ which joins P to the line of intersection of U, V. (It follows incidentally that W, W′ must separate U, V harmonically). Since $C - C' = W'^2$, it follows that C, C′ have contact along two generators.

The usual definition of a focus of a conicoid (see for instance, Bell's *Co-ordinate Geometry*, page 163) leads to the fact that P is a focus of a conicoid S if the circular cone with vertex P ( = point-sphere P) has two common plane sections with S. The other definition is that P is a focus of S if the enveloping cone from P to S is right circular, *i.e.*, has contact along two generators with the circular cone having its vertex at P. Lemmas (1) and (2) shew the equivalence of the two definitions of focus.

R. VAIDYANATHASWAMY.

## The Foci of a Conic.

The following method of obtaining the foci of a conic, though straight-forward, does not appear to find a place in the text-books commonly in use.

Let $P(x_1, y_1)$ be a focus of

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and I, J the circular points at infinity; then PI, PJ are tangents to the conic.

The tangential equations of P and of the conic are respectively

$$lx_1 + my_1 + 1 = 0$$

and $$Al^2 + 2Hlm + Bm^2 + 2Gl + 2Fm + C = 0.$$

Making the second equation homogeneous with the help of the first, we get

$$(A - 2Gx_1 + Cx_1^2)\,l^2 + 2\,(H - Gy_1 - Fx_1 + Cx_1y_1)\,lm + (B - 2\,Fy_1 + Cy_1^2)\,m^2 = 0 \quad \ldots \quad (1)$$

which we know gives us the two points at infinity on the two tangents that could be drawn to $S = 0$ from the point P. Hence if P is a focus of $S = 0$, the two points given by (1) must be the points I, J which are also given by

$$l^2 + m^2 = 0.$$

Hence, comparing co-efficients we have

$$C\,(x_1^2 - y_1^2) - 2Gx_1 + 2Fy_1 + A - B = \text{co-efft. of } l^2 - \text{co-efft. of } m^2 = 0.$$

$$Cx_1y_1 - Fx_1 - Gy_1 + H = \text{co-efft. of } lm = 0$$

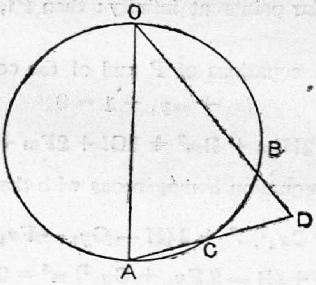
which are the equations of two conics through the foci.

If $C = 0$, then the conic $S = 0$ is a parabola and the two equations give us the co-ordinates of the real focus.

Sir Parashurambhau College, Poona. GOPALKRISHNA L. POPHALE.

## An approximate Construction for an Angle of 40°.

Let OA be a diameter of a circle and B an extremity of the perpendicular diameter. Take C on the arc AB so that BC = radius of the circle. Join AC and produce it to D making AD = AB. Then AOD is an angle nearly equal to 40°.




If AD = $a$ and angle AOD is denoted by $\theta$, then from the triangle AOD in which AO = $a\sqrt{2}$ and angle OAD is 75°, we have

$$\frac{a\sqrt{2}}{\sin(75^\circ+\theta)}=\frac{a}{\sin\theta}$$

$$\cot\theta=\frac{\sqrt{2}-\cos 75^\circ}{\sin 75^\circ}$$

whence $\theta = 39^\circ\ 54'$ approximately.

VISHNU MAHADEO TELANG.

# Solutions.

## Question 1325.

(S. M. SHAH, M.A.):—Sum the series

$$\sum_{n=0}^{n=\infty} b^n e^{-\frac{(z+2na)^2}{c}}$$

where $a$, $b$, $c$ are constants.

*Solution by S. D. Chowla.*

Call the series $f(z)$. Then

$$f(z) = e^{-\frac{z^2}{c}} \sum_0^\infty e^{-\frac{4n^2a^2}{c}} e^{n\left(\log b - \frac{4az}{c}\right)}.$$

If $a$, $b$, $c$, $z$ are real quantities this series can be evaluated in finite terms, only when

(i) $$z = \frac{c \log b}{4a}$$

and (ii) $\dfrac{4a^2c}{\pi} = \text{K}, \text{K}\sqrt{2}$ or $\text{K}\sqrt{3}$ where K is a rational number.

For, when these conditions are satisfied,

$$f(z) = e^{-\frac{z^2}{c}} \left\{ \sum_0^\infty \cdot q^{n^2 s} \right\} \qquad \text{where } q = e^{-\pi}$$

and $s$ stands for K, K $\sqrt{2}$ or K $\sqrt{3}$.

With the usual notation in Elliptic Functions we know that

$$\sum_0^\infty q^{n^2} = \cdot \frac{\sqrt{\frac{2\text{K}}{\pi}} + 1}{2}.$$

Again K $\left(= \int_0^{\frac{\pi}{2}} \dfrac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}\right)$ has been expressed finitely in terms of $\Gamma$-functions when $\dfrac{\text{K}'}{\text{K}} = 1, \sqrt{2}$ or $\sqrt{3}$. Hence it follows from

the theory of modular equations that K can be expressed finitely in terms of Γ-functions when the modulus $k$ satisfies $\frac{K'}{K} = l\sqrt{1}, l\sqrt{2}$ or $l\sqrt{3}$ where $l$ is a rational number.

In particular if

I. $b = 1,\ z = 0,\ c = 1,\ a = \frac{\sqrt{\pi}}{2}$,

then

$$f(z) = \sum_0^\infty e^{-n^2\pi} = \frac{\sqrt{\frac{2}{\pi}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}} + 1}{2}.$$

But
$$\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{\pi}}.$$

Therefore in this case

$$f(z) = \sum_0^\infty e^{-n^2\pi} = \frac{\sqrt{\frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}}} + 1}{2} = \frac{\Gamma(\frac{1}{4})}{2\sqrt[4]{2}\,\pi^{3/4}} + \frac{1}{2}.$$

II. $b = 1,\ z = 0,\ c = 1,\ a = \frac{\sqrt{\pi}}{2}\cdot\sqrt[4]{3}$

then
$$f(z) = \sum_0^\infty e^{-\sqrt{3}\,n^2\pi}.$$

But Legendre has proved that when

$$\frac{K'}{K} = \sqrt{3},\ \text{then}\ k = \sin 15°.$$

Hence
$$f(z) = \frac{\sqrt{\frac{2}{\pi}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2 15°\sin^2\phi}}} + 1}{2}.$$

And, we know,

$$\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2 15°\sin^2\phi}} = \frac{1}{2}\cdot\frac{1}{3^{3/4}}\cdot\frac{\Gamma(\frac{1}{3})\sqrt{\pi}}{\Gamma(\frac{2}{3})}.$$

Hence, in this case also $f(z)$ is expressed finitely in terms of Γ-functions.

## Question 1423.

(V. RAMASWAMY AIYAR) :—ABCD is a spherical quadrilateral whose sides are supposed rigid and jointed at the corners and in which the sums of the opposite sides are equal. If one of the sides be held fixed and the quadrilateral deformed on the sphere, show that the locus of the in-centre of the quadrilateral is a sphero-conic whose minor axis subtends a right angle at every point on the curve, the length of this axis being

$$\tan^{-1}\left[\frac{2}{\sin s}\sqrt{\sin a \sin b \sin c \sin d}\right]$$

where $a$, $b$, $c$, $d$ are the sides and $s$ is the semi-perimeter of the quadrilateral.

### *Solution by T. R. Raghava Sastri.*

Let ABCD be one position of the quadrilateral, the side AB being fixed. Let O be its in-centre and AB, BC, CD, DA, be $a$, $b$, $c$, $d$ respectively. On the side AB measure arcs AD′ and BC′ equal to $d$ and $b$ respectively. Then evidently the $\triangle$s AOD′ and AOD are congruent as also the $\triangle$s BOC′ and BOC . ... (A)

$\therefore$ The $\triangle$s COD and C′OD′ are congruent, the three sides of the one being equal to the three sides of the other ... (1)

Now $\angle AOD' + \angle BOC' = \angle AOB + \angle COD$

$= \angle AOB + \angle C'OD'$ from (1)

But $\angle AOD' + \angle BOC' = \angle AOD + \angle BOC$

whence $\angle AOB + \angle C'O'D' = \pi$.

Hence O is such that the known segments AB and C′D′ on the given great circle subtend supplementary angles at O. Hence the locus of O is easily proved (much in the same way as in plane geometry) to be such that the double points X′, Y of the involution formed by (A, C′) and (D′, B) subtend at every point of the locus a right angle.

*To show that it is a sphero-conic*, if M is the centre of the sphere take the plane MAB as the $z$ plane, M as the origin and the

straight line through M bisecting the arc XY as the $x$-axis. Then the locus of O is seen to be the cone $z^2 (1 - m^2) + y^2 - m^2 x^2 = 0$ where $m$ is the tangent of the angle MX or MY makes with the $x$-axis Or the equation of the sphero conic with the great circle XY as the $z$-axis and the great circle bisecting XY as the $y$-axis is

$$\frac{\sin^2 z}{m^2} + \frac{\sin^2 y\,(1 - m^2)}{m^2} = 1$$

which clearly shows that XY is the minor axis of the sphero-conic.

*To calculate the arc* XY : from the figure, OX and OY are the common bisectors of the angles AOC′ and D′OB. In the △ AOC′, AC′ = $a - b$ and if L be the mid. point of AC′

then
$$\frac{\tan LC'}{\tan LY} = \frac{\sin AO - \sin C'O}{\sin AO + \sin C'O}$$

and
$$\frac{\tan LC'}{\tan LX} = \frac{\sin AO + \sin C'O}{\sin AO - \sin C'O}$$

and remembering that C′O = CO from (A), we get from the above equalities

$$\tan XY = 2\,\frac{\sin (a - b) \sin AO \,.\, \sin CO}{\sin^2 AO - \sin^2 CO} \quad \ldots \quad (2)$$

If $r$ is the radius of the in-circle, from the triangle AOB

$$\sin AO \,.\, \sin BO \,.\, \sin A\hat{O}B = \sin a \sin r.$$

Similarly from triangle COD

$$\sin CO \,.\, \sin OD \,.\, \sin C\hat{O}D = \sin C \sin r.$$

But angles AOB and COD are supplementary. Hence by division

$$\frac{\sin AO \,.\, \sin BO}{\sin a} = \frac{\sin CO \,.\, \sin DO}{\sin C} \quad \ldots \quad (3)$$

Similarly
$$\frac{\sin BO \,.\, \sin CO}{\sin b} = \frac{\sin AO \,.\, \sin DO}{\sin d} \quad \ldots \quad (4)$$

From (3) and (4) we get easily

$$\frac{\sin AO . \sin CO}{\sin^2 AO - \sin^2 CO} = \frac{\sqrt{\sin a \sin b \sin c \sin d}}{\sin s \sin (a-b)}.$$

From (2) therefore we get the required value for tan XY.

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## Question 1470.

(A. NARASINGA RAO):—Given two equal circles, show that for every triangle inscribed in the one, there is a triangle inscribed in the other which has with it a common Steiner tricusp.

*Solution by T. R. Raghava Sastri.*

Take any triangle inscribed in the first circle, O, say. Then the nine-point centre of this triangle must be the nine-point centre of a certain triangle inscribed in the second equal circle O′. Therefore the ortho-centre of the latter triangle is also known as its nine-point and circum-centres are given. From this ortho-centre draw tangents to the Steiner tricusp of the first triangle. Then the triangle formed by the perpendicular tangents will be the required triangle. The proof is easily seen to follow from the question worked out on page 117, Part II of the Journal for October 1929.

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## Question 1509.

(S. D. CHOWLA):—If $-\frac{\pi}{7} < x < \frac{\pi}{7}$, prove that

$$\sum_1^{\infty} \frac{a_n \cos nx}{n} = 0,$$

where $a_1 = a_6 = a_9 = a_{10} = a_{11} = a_{12} = +1.$

$a_2 = a_3 = a_4 = a_5 = a_8 = a_{13} = -1$

$a_7 = a_{14} = 0, \ a_n = a_{n+14}.$

*Solution by S. Ramaswamy Aiyar.*

The given series is the real part of $S = \sum \frac{a_n e^{inx}}{n}$.

Let $f(x) = \log_e(1 + e^{ix})$, $f_r = f(x + r\phi)$, $\phi = \frac{2\pi}{7}$ and $\omega = e^{i\phi}$; then

$$f_0 = e^{ix} - \frac{e^{2ix}}{2} + \frac{e^{3ix}}{3} - \dots$$

$$f_1 = \omega e^{ix} - \frac{\omega^2}{2} e^{2ix} + \frac{\omega^3}{3} e^{3ix} - \dots$$

$$f_2 = \omega^2 e^{ix} - \frac{\omega^4}{2} e^{2ix} + \frac{\omega^6}{3} e^{3ix} - \dots, \text{ etc.}$$

... ... ... ... ... ...

From these, and from the given values of $a_n$,

$$f_0 + \omega^6 f_1 + \omega^5 f_2 + \omega^4 f_3 + \omega^3 f_4 + \omega^2 f_5 + \omega f_6$$
$$= 7 \times (\text{sum of the 1st, 8th, 15th, etc., terms of S}).$$

$$f_0 + \omega^5 f_1 + \omega^3 f_2 + \omega f_3 + \omega^6 f_4 + \omega^4 f_5 + \omega^2 f_6$$
$$= 7 \times (\text{sum of the 2nd, 9th, etc., terms of S}),$$

$$-(f_0 + \omega^4 f_1 + \omega f_2 + \omega^5 f_3 + \omega^2 f_4 + \omega^6 f_5 + \omega^3 f_6)$$
$$= 7 \times (\text{sum of the 3rd, 10th, etc., terms of S})$$

and so on.

Lastly

$$-(f_0 + \omega f_1 + \omega^2 f_2 + \omega^3 f_3 + \omega^4 f_4 + \omega^5 f_5 + \omega^6 f_6)$$
$$= 7 \times (\text{sum of the 6th, 13th, etc., terms of S})$$

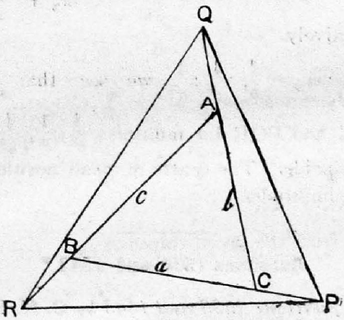
Adding

$$7S = (\omega^6 + \omega^5 + \omega^3 - \omega^4 - \omega^2 - \omega)$$
$$\times (f_1 + f_2 - f_3 + f_4 - f_5 - f_6).$$

It is easily verified that the first factor is purely imaginary and the second, real. Hence the real part of S is zero.

## Question 1517.

(S. N. MALURKAR) :—If the sides AB, BC, CA of a triangle be produced in order to P, Q, R such that the perpendicular from P on CA, Q on AB, R on BC are proportional to CA, AB, BC respectively, show that the triangle ABC, PQR are similar and find the centre of similitude.




*Solution by S. Ramaswamy Iyer, M.A.*

By hypothesis PC sin C, QA sin A, and RB sin B are proportional to $b$, $c$ and $a$ respectively so that we may put

$$PC = b\lambda/\sin C = \frac{2R\lambda b}{c} = \frac{kb}{c} \text{ say:}$$

$$QA = k\frac{c}{a} \text{ and similarly } RB = \frac{ka}{b}.$$

R being the circum-radius of ABC and $k$ a constant.

Now $\qquad PQ^2 = PC^2 + CQ^2 + 2PC.CQ \cos ACB,$

and substituting the values of PC, CQ, we obtain

$$\frac{PQ^2}{b^2} = k^2 \sum \frac{1}{a^2} + \frac{k \Sigma a^2}{abc} + 1.$$

The symmetry of the expression shows that $\frac{QR^2}{c^2}$ and $\frac{RP^2}{a^2}$ are equal to the same expression.

$\therefore$ $\Delta$ PQR and $\Delta$ABC are similar,

It is easily worked out that masses $m_1$, $m_2$ and $m_3$ at PQR are statically equivalent to masses

$$m_1 + \frac{2R\lambda}{abc}(m_1 b^2 - m_2 c^2);\quad m^2 + \frac{2R\lambda}{abc}(m_2 c^2 - m_3 a^2);\ \text{and}$$

$$m_3 + \frac{2R\lambda}{abc}(m_3 a^2 - m_1 b^2)$$

at C, A, B respectively.

$\therefore$ If $m_1 : m_2 : m_3 = \frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}$ we see that the centroids of the triangles ABC and PQR for multiples $\frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2}$ at C, A and B or at P, Q and R coincide. The centre of mean position for these masses is their centre of similitude.

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## Questions 1536 and 1543.*

*Remarks on Questions 1536 and 1543 by C. N. Srinivasiengar.*

(*a*) The last three lines on p. *219*, J. I. M. S., Part II, Vol. XIX, refer to a clause at the end of Question 1536, which has been omitted, *viz.*, "which do not meet all the generators of the surface," meaning that the parabolic curve cannot be a directrix (or singular generator). [ There may, however, be a finite number of generators which meet a given parabolic generator. The points of intersection should necessarily be either multiple points, or points whose indicatrices are cubics or higher curves which we shall refer to as cubical points].

(*b*) ***A correction*** : Line 6 from bottom on p. *219* referred to above. Add after 'double line,' the words " or a line of cubical points."

(*c*) . For more information about and a generalisation of Question 1536 (1), *Vide* J. I. M. S., Vol. XIX, Part I, p. 46.

(*d*) The result of Question 1536 (2) will be found in Salmon's *Analytical Geometry of Three Dimensions*, Vol. II, Art. 619, where he

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* Solved on p 220, Pt. II, Vol. XIX, No. 9, J.I.M.S.

gives the further property that any such parabolic generator is a torsal one. Salmon's method of proof is however very different. The method of curvilinear co-ordinates gives simpler proofs.

(*e*) Salmon enunciates the results thus: "The Hessian of a ruled surface meets the surface only in its multiple curves and in the generators each of which is intersected by one consecutive."

This statement appears to me to be incorret in part, for while it is a fact that in general a scroll possesses only a finite number of cubical points, one on each torsal generator (J. I. M. S., Vol. XIX, p. 47), it is conceivable that in special cases there may be a locus of such points Such a locus lies on the Hessian (*vide* J. I. M. S., Vol. XVIII, p. 194). If such a locus exists, it possesses the property of being a torsal generator also. The *necessary* conditions for such a locus are that a factor of the form $p - \alpha$, (where $\alpha$ is a constant) should be a factor of all the expressions $\psi''$, $\psi' f'' - \chi''$, $\chi' f'' - f' \chi''$, $\chi' - f' \psi'$ (see p. 219). One simple set of values for $f$, $\psi$, $\chi$ satisfying these conditions is given by

$$f = \frac{p^2}{2} - \alpha p,$$

$$\psi = \frac{p^3}{6} - \frac{p^2}{2} \alpha,$$

$$\chi = -\frac{p^3}{8} + \frac{p^3 \alpha}{6}.$$

The corresponding ruled surface has the equation

$$4(5x^2z + 6yz - 6\alpha y - x^2\alpha)(5z^3 - 8\alpha z^2 + 4\alpha^2 z + 4xz - 2\alpha x - \alpha^3)$$
$$= (10xz^2 - 6\alpha xz + 3x^2 + 2\alpha^2 x - 6y)^2.$$

The line, $p = \alpha$

*i.e.*, $$x = \alpha z - \frac{\alpha^2}{2},\ y = -\frac{\alpha^3}{3} z + \frac{\alpha^4}{24}$$

should therefore be either a double line or a line of cubical points for this surface, since $L = 0$, $M = 0$, $N = 0$ along it. That it is not a double line is verified, for differentiating with respect to $y$, there occurs a term

in $z^4$ which does not cancel identically, $x$ and $y$ being linear functions of $z$. We have thus an example of a ruled surface with a line of cubical points, and the Hessian contains this line.

(*f*) For similar reasons, the words 'in general' should be added to the second part of Question 1543.

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## Question 1548.

(S. NARAYANA AIYAR, M.A.):—Establish the following relations:—

(1) $$\frac{\text{sn } 3u}{\text{sn } u \text{ cn } 2u + \text{sn } 2u \text{ cn } u} = \frac{1 + \text{dn } 3u}{\text{dn } u + \text{dn } 2u}.$$

(2) $$\frac{\text{sn } 3u}{\text{sn } u \text{ dn } 2u + \text{sn } 2u \text{ dn } u} = \frac{1 + \text{cn } 3u}{\text{cn } u + \text{cn } 2u}.$$

### *Solution by Hukam Chand.*

These relations are particular cases of the following two relations:—

(A) $$\frac{\text{sn }(V_1 + V_2)}{s_1 c_2 + s_2 c_1} = \frac{1 + \text{dn }(V_1 + V_2)}{d_1 + d_2},$$

(B) $$\frac{\text{sn }(V_1 + V_2)}{s_1 d_2 + s_2 d_1} = \frac{1 + \text{cn }(V_1 + V_2)}{c_1 + c_2},$$

where, as usual $s_k$, $c_k$, $d_k$ denote respectively sn $V_k$, cn $V_k$, dn $V_k$.

To prove these relations we start with the identity

$$\text{cn } \alpha = \text{cn } \beta \text{ cn}(\beta - \alpha) + \text{sn } \beta \text{ sn}(\beta - \alpha) \text{ dn } \alpha^* \quad \ldots \quad (1)$$

which can easily be transformed into

$$\text{sn } \alpha \text{ dn}(\alpha - \beta) = \text{cn } \beta \text{ dn } \alpha \text{ sn}(\alpha - \beta) + \text{sn } \beta \text{ cn}(\alpha - \beta) \quad (2)$$

by writing $\alpha + K$ for $\alpha$.

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* *Fonctions Elliptiques*, Appeal and Lacour, p. 186; or Greenhill's *Elliptic Functions*, p. 116.

In relation (2) put $\alpha = V_1 + V_2$, $\beta = V_2$ and $\alpha = V_1 + V_2$, $\beta = V_1$ respectively to get

$$d_1 \operatorname{sn}(V_1 + V_2) = c_2 s_1 \operatorname{dn}(V_1 + V_2) + c_1 s_2 \quad \ldots \quad (3)$$

$$d_2 \operatorname{sn}(V_1 + V_2) = c_1 s_2 \operatorname{dn}(V_1 + V_2) + s_1 c_2 \quad \ldots \quad (4)$$

Adding (3) and (4) we get relation A.

Again in (2) put $\beta = V_1 + V_2$, $\alpha = V_2$ and $\beta = V_1 + V_2$, $\alpha = V$ respectively, to get

$$s_2 d_1 \quad - s_1 d_2 \operatorname{cn}(V_1 + V_2) + \operatorname{sn}(V_1 + V_2) c_1 \quad \ldots \quad (5)$$

$$s\, d_2 = - s_2 d_1 \operatorname{cn}(V_1 + V_2) + \operatorname{sn}(V_1 + V_2) c_2 \quad \ldots \quad (6)$$

Adding (5) and (6) we get relation B.

It may be noted that on eliminating $\operatorname{dn}(V_1 + V_2)$ from (3) and (4) or $\operatorname{cn}(V_1 + V_2)$ from (5) and (6), we get

$$\operatorname{sn}(V_1 + V_2) = \frac{c_1^2 - c_2^2}{s_2 c_1 d_1 - s_1 c_2 d_2}$$

which is an addition formula for $\operatorname{sn}(V_1 + V_2)$.

If we eliminate $\operatorname{sn}(V_1 + V_2)$ from (3) and (4), (5) and (6), we get the addition formulae for $\operatorname{dn}(V_1 + V_2)$ and $\operatorname{cn}(V_1 + V_2)$, *viz.*,

$$\operatorname{dn}(V_1 + V_2) = \frac{s_1 d_1 c_2 - s_2 d_2 c_1}{s_1 c_2 d_2 - s_2 c_1 d_1}$$

$$\operatorname{cn}(V_1 + V_2) = \frac{s_2 c_2 d_1 - s_1 c_1 d_2}{s_2 d_1 c_1 - s_1 d_2 c_2}.$$

In relation (2) replace $\beta$ by $-\beta$, we then get

$$\operatorname{sn}\alpha \operatorname{dn}(\alpha + \beta) = \operatorname{cn}\beta \operatorname{sn}(\alpha + \beta) \operatorname{dn}\alpha - \operatorname{sn}\beta \operatorname{cn}(\alpha + \beta).$$

Replace $\alpha$ by $V_1$ and $\beta$ by $V_2 + iK'$, we then have

$$\operatorname{dn}(V_1 + V_2) = d_1 d_2 - k^2 s_1 s_2 \operatorname{cn}(V_1 + V_2) \quad \ldots \quad (7)$$

a relation proved by Greenhill by Spherical Trigonometry (cf. page 131, *Elliptic Functions*).

Again, in (1) put $\alpha = V_1 + V_2$, $\beta = V_2$

$$\text{cn}(V_1 + V_2) = c_1 c_2 - s_1 s_2 \,\text{dn}(V_1 + V_2). \qquad \dots \quad (8)$$

Relations (7) and (8) when solved for $\text{cn}(V_1 + V_2)$ and $\text{dn}(V_1 + V_2)$ give us other expressions for the addition formulae for $\text{cn}(V_1 + V_2)$ and $\text{dn}(V_1 + V_2)$, *viz.*,

$$\text{cn}(V_1 + V_2) = (c_1 c_2 - k^2 s_1 s_2 d_1 d_2) / D$$

$$\text{dn}(V_1 + V_2) = (d_1 d_2 - k^2 s_1 s_2 c_1 c_2) / D$$

where $$D = 1 - k^2 s_1^2 s_2^2.$$

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## Question 1554.

(C. N. SRINIVASIENGAR) :—(1) Prove that the surface defined by the arc-element $ds^2 = 2f(p)\,dp\,dq$ is a developable having two systems of plane lines of curvature.

(2) Prove that the surface defined by the values of the fundamental magnitudes

$$E = 0,\ F = f(p),\ G = 0,\ L = \frac{a^2}{b}\{f(p)\}^2,\ M = af(p),\ N = b$$

(where $a$ and $b$ are any constants) is a right circular cylinder whose radius of cross-section is equal to $(2a)^{-1}$.

### *Solution by the Propcser.*

(1) The surface is referred to the null-lines as parametric curves. The Gaussian curvature at any point of the surface is given by

$$-\frac{1}{F}\frac{\partial^2}{\partial p \partial q}(\log F)$$

[*vide* Forsyth's *Differential Geometry*, § 56]. When $F = f(p)$, the surface is therefore a developable.

The equation to the lines of curvature reduces in this case to $L\,dp^2 - N\,dq^2 = 0$, *i.e.*, $\sqrt{L}\,dp + \sqrt{N}\,dq = 0$, and $\sqrt{L}\,dp - \sqrt{N}\,dq = 0$. One of these two equations must give the generators of the developable. Any generator is a geodesic. The differential equation of geodesics reduces in the present case to

$$\frac{d^2 q}{dp^2} = \frac{f'(p)}{f(p)}\frac{dq}{dp}.$$

Clearly, if this equation is satisfied when we put $\frac{dq}{dp}$ equal to one of the two expressions

$$+\sqrt{\frac{L}{N}},\ -\sqrt{\frac{L}{N}},$$

it is also satisfied by the other value.

Therefore, any line of curvature of the surface is a geodesic, and hence by a well-known result, a plane curve.

(2) It may be verified that the given values of the fundamental magnitudes satisfy Gauss's characteristic equation and the Mainardi-Codazzi relations. A definite surface is therefore defined save as to its position and orientation in space.

By (1), the surface is a developable having two systems of plane lines of curvature.

Now the non-vanishing principal curvature at any point of the surface is

$$\frac{EN + GL - 2FM}{EG - F^2} = 2a.$$

The principal radius of curvature is thus constant all along the surface, and equal to $1/2a$.

The only surface which possesses all these properties is the right-circular cylinder.

# Questions for Solution.

*Proposers of Questions are requested to send their own solutions along with their questions.*

**1624.** (K. J. SANJANA, M.A.) :—Give a strict demonstration of the following theorem which is the converse of a well known theorem of Fermat.

The squares of two numbers, which are prime to each other, are added together : prove that the sum is, except in one obvious case, a prime number of the form 4M + 1 or a product of prime numbers of this form. The exception occurs when the numbers squared are both odd, in which case the sum contains the factor 2 which, though a prime number, is not of the form 4M + 1 ; when this factor is removed the remaining number has the property enunciated above.

**1625.** (K. J. SANJANA and K. F. VAKIL) :—Resolve each of the following numbers into sums of two squares in all possible ways :

(1) 600,517 ; (2) 9,637,417 ; (3) 9,997,481 ;

(4) 18,835,405 ; (5) 65,536,010.

**1626.** (K. J. SANJANA and K. F. VAKIL) :—Decompose in all possible way the number 1197,721,083 into the sum of three square numbers in Arithmetic Progression.

**1627.** (R. VAIDYANATHASWAMY) : —A surface S, supposed rigid, is moved by an infinitesimal rotation about an axis L into the position S′. If S′ touches S at a point O, prove that L must lie in one of the principal normal sections at O, and pass through the centre of curvature of the other principal normal section.

**1628.** (S. CHOWLA) : —(1) If $\sigma(n)$ denotes the sum of the divisors of $n$, show that

$$\sum_{n \leqslant \sqrt{x}} \sigma(x - n^2) = O(x^{3/2}).$$

(2) If $a$, $b$, $c$ are positive integers, show that

$$\sum_{ax^2+by^2+cz^2=n} 1 = O(\sqrt{n}\,\log^2 n).$$

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2 American Journal of Mathematics, **54**, 3 (3 copies).
3 American Mathematical Monthly, **39**, 5, 6 (3 copies).
4 Annales de l'Ecole Normale Superieure, 4, 5, 6, 1932 (2 copies each).
5 Astrophysical Journal, **75**, 3, 4.
6 Bulletin of the American Mathematical Society, **38**, 5.
7 Bulletin of the Calcutta Mathematical Society, **24**, 1.
8 Bulletin des Sciences Mathematiques, **56**, April & May 1932.
9 Crelle's Journal, **166**, 4, **168**, 1.
10 Mathematical Gazette, **16**, 218, 219 (3 copies each).
11 Mathematische Annalen, **106**, 4
12 Monthly Notices of the Royal Astronomical Society, **92**, 6.
13 Philosophical Transactions of the Royal Society, **231**, pp. 29—53.
14 Popular Astronomy, **40**, 5, 6.
15 Proceedings of the Cambridge Philosophical Society, **28**, 2.
16 Proceedings of the London Mathematical Society, **33**, 7 (3 copies).
17 Proceedings of the Physico-Mathematical Society of Japan, **14**, 3, 4, 5, 6.
18 Proceedings of the Royal Society of London, **136**, 829, 830, 831.
19 Quarterly Journal of Mathematics, **3**, 10.
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22 Revista Mathematica Hispano Americano, **7**, 1, 2 (2 copies).

## Books.

1 A Determination of the types of Plane Cremona Transformations with not more than 9 F-points, by Mildred Ellen Taylor.
2 On Identities arising from Solutions of $q$-Difference Equations and some Interpretations in Number Theory, by George William Starcher.
3 On complete systems under certain finite groups, by Carl Walther Strom.
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